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Notes on Leibniz' *De Arte Combinatoria*

The purpose of these notes is to present a logistic system (syntax, semantics, and proof theory) that "reconstructs" the sort Leibniz had in mind in thinking about truth and necessity. The ideas developed here are based on the sketch provided earlier in his career (at age 18) by Leibniz in his essay *De Arte Combinatora* of his ideas on conceptual construction, analytic truth, and proof as finite reduction to identity statements. I have however extrapolated from this early material to incorporate infinite concepts, existence, God, positive and negative properties and explicit analyses of truth and necessity as these ideas were developed in his later work.

Syntax. The syntax begins by positing a set of basic terms that stand for primitive ideas:

First Terms: $t_1,...,t_n$. Among the first terms is *exists*.

Primitive terms may be joined together to make longer terms. In principle some of these longer terms may be infinitely long, though those of finite length are special. To define strings of first terms we make use of the concatenation operation: let $x^\frown y$ mean the result of writing (*concatenating*) x and y. (Later when there is no possibility of confusion, we shall suppress the concatenation symbol and refer to $a^\frown b^\frown c^\frown d$ as abcd.)

Finite Terms: If $t^1_1$and t^1_2, are first terms, then $t^1_1{}^\frown t^1_2$ is a finite term.
If t^n_i is a finite term and t^1_j is a first term, then $t^n_i{}^\frown t^1_j$ is a finite term.
Nothing else is a finite term.

Infinite Terms: any countably infinite subset of First Terms.
Among the infinite terms is *God*.

Terms: the union Finite Terms and Infinite Terms.

Leibniz introduces a special vocabulary for discussing finite terms:

Terms of conXnation (defined inductively):
If t^1_i is a first term,
then t^1_i is a term of con1nation with exponent 1 and rank i.
If t^n_i is a term of conNnation and t^1_j is a term of con1nation,
then $t^n_i{}^\frown t^1_j$ a term of conN+1nation,
with exponent n+1,
and a rank that is determined by three factors:
the ranks of t^n_i, t^i_j, and the ranks of those terms of conN+1nation that have a lesser rank than $t^n_i{}^\frown t^1_j$.
Nothing else is a term of conXnation.

Clearly the set of all terms of conXnation for some x is identical to the set *Finite Terms*. We let t^n_i refer to the term of conJnation of rank i.

Fraction notation: if t^{n+1}_k is some term $t^n_i \frown t^1_j$ of conN+1nation, then another name for t^{n+1}_k is $<i/n,t^1_j>$.

We shall adopt some special notation for infinite terms. If $\{t^1_1,....,t^1_n,...\}$ is an infinite term (a set of first terms) we shall refer it briefly as $\{t\}_i$. A *proposition* is any expression t **is** t' such that t and t' are terms. It is permitted that these terms be infinite. A *finite proposition* is any: t^i ***is*** t^j, such that t^i is a term of conInation and t^j is a term of conJnation, for natural numbers i and j. An infinite propositions is any t^i ***is*** t^j such that both t^i and t^j are either finite or infinite terms and at least one of t^i and t^j is infinite. Notice that it follows from the definitions that though there are a finite number of first terms, there are an infinite number of finite terms and of finite propositions. A proposition that is not finite is said to be *infinite*. Such propositions will contain at least one infinite term.

Intensional Semantics

Conceptual Structure. For an intensional semantics we posit a set $\mathcal{C}$ of concepts for which there is a binary inclusion relation ≤ and a binary operation + of concept composition or addition. In modern metalogic the way to do this is to specify the relevant sort of "structure" understood as an abstract structure with certain specified structure features governing $\mathcal{C}$, ≤ and +. We also distinguish between positive and negative concepts and add concepts of existence and God. By *a Leibnizian intensional structure* is meant any structure $<\mathcal{C},\leq,+,\mathcal{G}>$ such that

1. $<\mathcal{C},\leq>$ is a partially ordered structure:
   ≤ is reflexive, transitive and anti-symmetric;
2. $<\mathcal{C},\wedge>$ is an infinite join semi-lattice determined by $<\mathcal{C},\leq>$:
   if $\mathcal{A}$ is an infinite subset of $\mathcal{C}$ (in which case we call $\mathcal{A}$ an *infinite concept*), then there is a least upper bound of $\mathcal{A}$ (briefly, a lub$\mathcal{A}$) in $\mathcal{C}$ (here the *least upper bound* of $\mathcal{A}$ in $\mathcal{C}$ is defined as the unique $z\in\mathcal{C}$ such that for any c in $\mathcal{A}$, $c\leq z$, and for any w, if for all c in $\mathcal{A}$, $c\leq w$, then $z\leq w$);
3. for any $c_1,...,c_n$ in $\mathcal{C}$, $c_1+...+c_n$ is defined as lub$\{c_1,...,c_n\}$,
   for any infinite subset $\mathcal{A}$ of $\mathcal{C}$, $+\mathcal{A}$ is defined as lub$\mathcal{A}$;
4. $\mathcal{G}$ (called the concept of *God*) is $+\mathcal{C}$

Theorem: If $<\mathcal{C},\leq,+,\mathcal{G}>$ is an intensional structure and let $c,d\in\mathcal{C}$, it follows that:

1. $c\leq d$ iff $c=c+d$,
2. $\mathcal{C}$ is closed under +, and + is idempotent, commutative, and associative,
3. if $\mathcal{A}$ is an infinite concept, then $c\in\mathcal{A}$ only if $c\leq+\mathcal{A}$,
4. $+\mathcal{C}$ is a supremum in $\mathcal{C}$ (i.e. for any $c\in\mathcal{C}$, $c\leq+\mathcal{C}$ and $+\mathcal{C}\in\mathcal{C}$);

Let a,b,c and d range over $\mathcal{C}$. It is also useful to have a notion of concept *subtraction.* Let c-d be defined as follows:

if c is a finite concept, c-d is that concept b such that d+b=c, if there is such a concept, and c-d is undefined otherwise;

if c is an infinite concept $\mathcal{A}$ then c-d is $\mathcal{A}$–{d}, i.e. it is the set theoretic relative complementation of $\mathcal{A}$ and {d} (i.e. c-d = {d|e∈c and e≠d}).

Theorem. For any c and c in $\mathcal{C}$, either c≤d or c≤+$\mathcal{C}$–d

Intensional Interpretations.[1] By an intensional interpretation we mean any assignment of concepts to terms that mirrors their internal structure. That is, an *intensional interpretation* is any function Int with domain Terms and range $\mathcal{C}$ such that:

1. If t_i is a first term (i.e. term of con1nation), then $Int(t_i) \in \mathcal{C}$.
2. If t_k is some term $t_i{}^\frown t^1{}_j$ of conN+1nation, then $Int(t_k)=Int(t_i)+Int(t^1{}_j)$.
3. If $\{t\}_i$ is some infinite term, Int $(\{t\}_i)=+\{Int(t^1)|\ t^1\in\{t\}_i\}$.
4. Int(*God*)=$\mathcal{G}$.

(Algebraically, an intensional interpretation Int is what is called a *homomorphism* from the grammatical structure <Terms, ⁀> to the conceptual structure < $\mathcal{C}$,+>.) Since Leibniz' languages are ideal, it is also plausible to require the stronger condition that the mapping Int be 1 to 1 (and hence an isomorphism), though since this extra condition plays no role here is will not be formally required.

Leibniz frequently identifies truth with conceptual inclusion. For some purposes it might be important to build the notion of an "atomic" concept into the definition of the intensional structure, but for our purposes here we shall refer to an *atomic concept* as any c in $\mathcal{C}$ that is the intension of some first term (i.e. such that for some first term t_i, $Int(t_i)=c$). Following modern usage, let us reserve the term analytic truth for this idea:

t_i ***is*** t_j is said to be *analytically true* for interpretation Int iff $Int(t_j)\leq Int(t_i)$.

Extensional Semantics (Possible Worlds)

Possible Worlds. In modern logic, possible worlds would be understood as extensional "models" that conform to the restrictions of a given intensional interpretation. Given the interpretation, a possible world will consist of an assignment of sets (extensions) to concepts (and hence to terms) in a manner that mirrors their internal structure. Let us define a *possible world* relative to an intensional interpretation Int to be any W that assigns "extensions" to concepts as follows: W is a function with domain $\mathcal{C}$ such that

1. If c is an atomic concept, then W(c) is some set D of possible objects ("the objets that fall under c in the world W");
2. If c is some concept a+b, then $W(c)=W(a)\cap W(b)$.
3. if c is some infinite concept $\mathcal{A}$, then $W(c)=\bigcap\{W(d)|d\leq\mathcal{A}\}$

Finally, the extensional interpretation of the syntax in a possible world W for Int assigns to a term the set determined by its concept and a truth-value to a proposition accordingly to whether the extension of the predicate embraces than of the subject. By the *extensional interpretation* Ext_W for the possible world W

[1] The terminology and basic semantic framework here is adapted from that of Rudolf Carnap, *Meaning and Necessity*, and Richard Montague, "Intensional Logic."

relative to intensional interpretation Int assigns extensions to terms and truth-values to propositions as follows:

1. If t_i is a term, $Ext_W(t_i)=W(Int(t_i))$;
2. If t_i ***is*** t_j is a proposition, $Ext_W(t_i$ ***is*** $t_j)$=T if $Ext_W(t_i) \subseteq Ext_W(t_j)$,
   $Ext(t_i$ ***is*** $t_j)$=F if not$(Ext_W(t_i) \subseteq Ext_W(t_j))$.

Logical Truth. Let a proposition P be called a *logical truth* relative to Int (briefly, $\models$P) iff, for all possible worlds W of Int, $Ext_W(P)$=T.

Theorem. 1. t_i ***is*** t_j is an analytic truth relative to Int iff it is a logical truth relative to Int.

2. If $Int(t_j) \leq Int(t_i)$, then for W relative to Int, $Ext_W(t_i) \subseteq Ext_W(t_j)$.

Remark. Leibniz allows for possible worlds to vary in "perfection, " and for the use of negations to describe privations of such perfection. These ideas are essentially neoplatonic. Logically they presuppose a ranking on "worlds" and a neoplatonic privative negation. Such theories may be developed coherently by imposing additional features to the syntax and semantic structure, but are not developed here because they play no role in the points to be made.

Proof Theory, Necessity and Contingency. Although Leibniz frequently says that all truth is conceptual inclusion, i.e. that truth is analytic truth, he also makes a distinction between necessary and contingent truths. Ordinarily in modern logic, necessary truth is identified with what we have called logical truth, and contingent truth with truth in a possible world. If all truths were analytic and necessary truth was the same logical truth, then truth and necessity collapse, and there could be no contingent truths. Leibniz avoids this problem by adopting what is now a non-standard notion of necessary truth. Leibniz defends what we would call today a proof theoretic concept of necessity by identifying necessity with provability. To do so Leibniz forges a distinction between truth defined semantically (e.g. analytic and logical truth) and a purely syntactically definable notion of a proposition's having a proof. He is arguably the first philosopher to do so clearly, and to complete the project we present here a version of his proof theory.

Proof Theory. Leibniz understands proofs to be syntactic derivations of propositions. They take what he calls "identity" propositions as axioms. Inferences progress by adding first terms to the subject of earlier propositions in the proof, or by subtracting first terms from the predicates of earlier lines. We begin by defining the set of axiom as the set of identity propositions axioms:

Basic Propositions (Axioms): any finite proposition of the form t_i ***is*** t_i.

(Also called *identity propositions*.)

Inferences proceed by adding and subtracting first terms to subjects and predicates respectively.

Inference Rule:

from $t_i{}^\frown t_j$ ***is*** t_k infer $t_i{}^\frown t^{1\frown} t_j$ ***is*** t_j;

from infer t_k *is* $ti^\frown t^{1\frown} t_j$ infer t_k ***is*** $t_i{}^\frown t_j$;

The process is complicated somewhat because Leibniz envisages language as containing abbreviations in which shorter expressions are used in place of long terms for which they are synonymous. As defined in the syntax, genuine terms (in the set Terms) are all finite concatenations of first terms. These expressions we shall say are in *primitive notation*. Let us now allow that such terms may be abbreviated by a single expression. Let a *defined term* be any expression E that is defined as abbreviating a term t_i (in Terms) by means of a definition of the form: E $=_{def}$ t_i. (For example we might have the definition: A $=_{def}$ abcd.) We draw together all definitions into a set that we call the *Lexicon*. Note that the Lexicon could be infinitely large. It is a standard rule in logic (and mathematics) that it is permissible to replace a term in any line of a proof by either its abbreviation (its *definiendum*) if it is a primitive term, or by its analysis into primitive notation (its *definiens*) if it is a defined term. Let P[t] be a proposition containing a term t and P[E] be like P[t] except for containing E at one or more places where P[t] contains t.

Rule of Definition: if E $=_{def}$ t_i, from P[t] infer P[E], and from P[E] infer P[t].

A proof may now be defined as any derivation from the axioms by the rules:

Proof: any finite series of propositions such that each is a basic proposition or follows by the inference rules (including the Rule of Definition) from previous members of the series.

Let us say a proposition P is (*finitely*) p*rovable* (alternative terminology is P is a *theorem*, is *necessary* or in symbols $\vdash$P) iff P is the last line of some proof.

Examples: Here are four proofs (read down each column). Let A $=_{def}$ abcd:

a is a	abcd is abcd	ab is ab	A is A
ab is a	abcd is abc	abc is ab	A is abcd
abc is a	abcd is ab	abcd is ab	A is abc
abcd is a	abcd is a	A is ab	A is ab

(Following Aristotle's usage in the *Prior Analytics*, Leibniz himself talks of “reductions” instead of "proofs". A reduction is just an upside down proof in which the first line is what is to be proved and you work down the page to the basic identity axiom.) Note that it follows from the definition of proof that all proofs have a finite number of lines. It is very important for Leibniz that necessity is *finitely* provable. *Contingent* propositions, he says, are ones that are true in his sense (i.e. analytically true) but for which there is no finite proof. The concept of God or of a possible world for Leibniz are infinite concepts and the term *God* abbreviates an infinite terms standing for an infinite concept.

Remark. Notice since infinite terms are literally infinite lists of basic terms, they are infinite in length and hence are precluded from appearance in a proof. Thus thought the following inference rules that employ infinite terms are valid, they are not proof theoretical acceptable:

from $\{t\}_i$ ***is*** t_k infer $\{t\}_i \cup \{t^1\}$ is t_k;
from t_k is $\{t\}_i$ infer t_k ***is*** $\{t\}_i - \{t^1\}$.

Theorem. The notion of proof is sound and complete for finite propositions, i.e. provability and logical (and hence analytic) truth coincide:

Finite Soundness:
if P is finite and $\vdash$P (equivalently, P is necessary),
then $\models$P (equivalently, P is analytic).

Finite Completeness:
if P is finite and $\models$P (equivalently, P is analytic),

then $\vdash$P (equivalently, P is necessary).

Theorem. If P is infinite, then not($\vdash$P)

Proof. Let P contain an infinite term $\{t\}_i$, and assume for a *reductio* that $\vdash$P. Then there is some proof of P. Moreover, if $\{t\}_i$ is the subject of P, there is for every first term t^1 in $\{t\}_i$ at line introducing that term to the subject. But then since there are an infinite number of such first terms in $\{t\}_i$ there an infinite number of lines in the proof. But a proof is only finitely long. Hence by *reductio.* There is no proof of P. The reasoning is similar if $\{t\}_i$ occurs as the predicate of P. Q.E.D.

Theorem. Soundness holds for both finite and infinite propositions, but completeness fails for infinite propositions:

Soundness:

For any P, if $\vdash$P, the $\models$P (equivalently, P is analytic)

Failure of Completeness: There is some infinite propositions P such that $\models$P(equivalently, P is analytic) but not($\vdash$P).

Questions:

1. If all truth is conceptual inclusion (analytic truth), is there any notion in Leibniz for “truth in a possible world” (modern day contingent truth)? (Perhaps adding the indexical modal operator *actually* would reintroduce the distinction.)

2. Is the proposition *God exists* true (i.e. analytic)? Is it provable? Is it necessary? (Prove your answer to each.) Is the constellation of answers odd? Explain.

For a partial English translation of the Latin text, see G.H.R.Parkinson, *Leibniz, Logical Papers* (Oxford: Clarendon Press, 1966). Recent work on Leibniz's logic:

Wolfgang Lenzen,
"On Leibniz's Essay '*Mathesis rationis*' (Critical Edition and Commentary)".
Topoi 9, 1990, 29-59.

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Berlin (de Gruyter), 1990 (Grundlagen der Kommunikation und Kognition).

--------, "Leibniz's Logic, " in vol. 2 of the *Handbook of the History and Philosophy of Logic* (John Woods, ed.), at http://www.philosophie.uni-osnabrueck.de/Woods.htm

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Forthcomming in Mentis (Paderborn).

Mugnai,Massimo, *Leibniz' Theory of Relations*

Swoyer,Chris, "Leibniz on Intension and Extension"
Nous, 29:1 (Mr 1995) pp. 96-114.

--------, "Leibniz's Calculus of Real Addition", *Studia Leibnitiana,* 26:1 (1994), pp. 1-30.

Appendix

The following is a partial Latin text published on the Internet (St.-Michaels-Gymnasium Metten, metten_gym@degnet.de)

Dissertatio de Arte Combinatoria,
cum Appendice.

1666.

...

Definitiones

1. Variatio h. l. est mutatio relationis. Mutatio enim alia substantiae est, alia quantitatis, alia qualitatis; alia nihil in re mutat, sed solum respectum, situm, conjunctionem cum alio aliquo.

2. Variabilitas est ipsa quantitatis omnium variationum. Termini enim potentiarum in abstracto sumti quantitatem earum denotant, ita enim in Mechanicis frequenter loquuntur, potentias machinarum duarum duplas esse invicem.

3. Situs est localitas partium.

4. Situs est vel absolutus vel relatus: ille partium cum toto, hic partium ad partes. In illo spectantur numerus locorum et distantia ab initio et fine, in hoc neque initium neque finis intelligitur, sed spectatur tantum distantia partis a data parte. Hinc ille exprimitur linea aut lineis figuram non claudentibus neque in se redeuntibus, et optime linea recta; hic linea aut lineis figuram claudentibus, et optime circulo. In illo prioritatis et posterioritatis ratio habetur maxima, in hoc nulla. Illum igitur optime Ordinem dixeris;

5. Hunc vicinitatem, illum dispositionem, hunc compostitionem. Igitur ratione ordinis differunt situs sequentes: abcd. bcda. cdab. dabc. At in Vicinitate nulla variatio, sed unus situs esse intelligitur, hic nempe:

b

a *c*

d

Unde festivissimus Taubermannus, cum Decanus Facultatis philosophiae esset, dicitur Witebergae in publico programmate seriem candidatorum Magisterii circulari dispositione complexus, ne avidi lectores intelligerent, quis suillum locum teneret.

6. Variabilitatem ordinis intelligemus fere, quando ponemus Variationes „κατ‘ εξοχην“ v. g. Res IV. possunt transponi modis 24.

7. Variabilitatem complexionis dicimus Complexiones. v. g. Res IV. modis diversis 15. invicem conjugi possunt.

8. Numerum rerum variandarum dicemus simpliciter, Numerum, v. g. IV. in casu proposito.

9. Complexio, est unio minoris totius in majori, uti in prooemio declaravimus.

10. Ut autem certa complexio determinetur, majus totum dividendum est in partes aequales sippositas ut minimas, (id est quae nunc quidem non ulterius dividantur) ex quibus componitur et quarum variatione variatur complexio seu totum minus; quia igitur totum ipsum minus, majus minusve est, prout plures partes una vice ingrediuntur; numerum simul ac semel conjungendarum partium, seu unitatum, dicemus Exponentem, exemplo progressionis geometricae, v. g. sit totum ABCD. Si tota minora constare debent ex 2. partibus, v. g. AB. AC. AD. BC. BD. CD. exponens erit 2. sin ex tribus, v. g. ABC. ABD. ACD. BCD. exponens erit 3.

11. Dato exponente complexiones ita scribemus: si exponens erit 2. Com2nationem (combinationem); si 3. Con3nationem (conternationem); si 4. Con4nationem, etc.

12. Complexiones simpliciter sunt omnes complexiones omnium exponentium computatae, v. g. 15. (de 4. Numero) quae componuntur ex 4. (Unione) 6. (com2natione) 4. (con3natione) 1. (con4natione).

13. Variatio utilis (inutilis), est quae propter materiam subjectam locum habere non potest; v. g. 4. Elementa com2nari possunt 6. modis; sed duae com2nationes sunt inutiles, nempe quibus contrariae Ignis, aqua, aër, terra com2nantur.

14. Classis rerum est totum minus, constans ex rebus convenientibus in certo tertio, tanquam partibus; sic tamen ut reliquae classes contineant res contradistinctas. v. g. infra probl. 3. ubi de classibus opinionum circa summum bonum ex B. Augustino agemus.

15. Caput Variationis est positio certarum partium; Forma variationis, omnium, quae in pluribus variationibus obtinet. v. infra probl. 7.

16. Variationes communes sunt in quibus plura capita concurrunt, v. infr. probl. 8 et 9.

17. Res homogena est quae est aeque dato loco ponibilis salvo capite. Monadica autem quae non habet homogeneam. v. probl. 7.

18. Caput multiplicabile dicetur, cujus partes possunt variari.

19. Res repetita est quae in eadem variatione saepius ponitur. v. probl. 6.

20. Signo + designamus additionem, - subtractionem, $\cap$ multiplicationem, $\cup$ divisionem, f. facit, seu summam, = aequalitatem. In prioribus duobus et ultimo convenimus cum Cartesio, Algebraistis, aliisque: Alia signa habet Isaacus Barrowius in sua editione Euclidis, Cantabrig. 8vo, anno 1655.

Problemata.

Tria sunt quae spectari debent: Problemata, Theoremata, usus; in singulis problematis usum adjecimus; sicubi operae pretium videbatur, et theotemata. Problematum autem quibusdam rationem solutionis addidimus. Ex iis partem posteriorem primi, secundum et quartum aliis debemus, reliqua ipsi eruimus. Quis illa primus detexerit eonem et Nicolaum Tartaleam extare dicit. In Cardani tamen practica Arithmetica quae pignoramus. Schwenterus Delic. 1. 1. Sect. 1. prop. 32. apud Hieronymum Cardanum, Johannem Butrodiit Mediolani anno 1539. nihil reperimus. Inprimis dilucide, quicquid dudum habetur, proposuit Christoph. Clavius in Com. supra Joh. de Sacro Bosco Sphaer. edit. Romae forma 4ta anno 1785 p. 33. seqq.

Probl. I.

Dato numero et exponente complexiones invenire.

Solutionis duo sunt modi, unus de omnibus complexionibus, alter de com2nationibus solum: ille quidem est generalior, hic vero pauciora requirit data, nempe numerum solum et exponentem; cum ille etiam praesupponat inventas complexiones antecedentes. Generaliorem modum nos deteximus, specialis est vulgatus. Solutio illius talis est: "„addantur complexiones exponentis antecedentis et dati de numero antecedenti, productum erunt complexiones quaesitae;“ v. g. est numerus datus 4, exponens datus 3. addantur de numero antecedente 3. com2nationes 3. est con3natio 1. (3+1. f. 4.) productum 4. erit quaesitum. Sed cum praerequirantur complexiones numeri antecedentis, construenda est tabula I. in qua linea suprema a sinistra dextrorsum continet Numeros, a 0 usque ad 12. utrimque inclusive, satis enim esse duximus huc usque progredi, quam facile est continuare: linea extrema sinistra a summo deorsum continet Exponentes a 0. ad 12. linea infima a sinistra dextrorsum continet Complexiones simpliciter. Reliquae inter has lineae continent complexiones dato numero qui sibi in vertice directe respondet, et exponente qui e regione sinistra. Ratio solutionis, et fundamentum tabulae patebit, si demonstraverimus, Complexiones dati numeri et exponentis oriri ex summa complexionum de numero praecedenti exponentis et praecedentis et dati. Sit enim numerus datus 5, exponens datus 3. Erit numerus antecedens 4. is habet con3nationes 4, per Tabulam I. com2nationes 6. Jam numerus 5. habet omnes con3nationes quas praecedens (in toto enim et pars continetur) nempe 4. et praeterea tot quot praecedens habet com2nationes, nova enim res qua numerus 5. excedit 4. addita singulis com2nationibus hujus, facit totidem novas con3nationes nempe 6.+ 4. f. 10. E. Complexiones dati numeri etc. Q.E.D.

Tabula I.

	0	1	1	1	1	1	1	1	1	1	1	1	1	1	
	1	0	1	2	3	4	5	6	n	8u	9m	10e	11r	12i	C
	2	0	0	1	3	6	10	15	21	28	36	45	55	66	o
E	3	0	0	0	1	4	10	20	35	56	84	120	165	220	m
x	4	0	0	0	0	1	5	15	35	70	126	210	330	495	p
p	5	0	0	0	0	0	1	6	21	56	126	252	462	792	l
o	6	0	0	0	0	0	0	1	7	28	84	210	462	924	e
n	7	0	0	0	0	0	0	0	1	8	36	120	330	792	x
e	8	0	0	0	0	0	0	0	0	1	9	45	165	495	i
n	9	0	0	0	0	0	0	0	0	0	1	10	55	220	o
t	10	0	0	0	0	0	0	0	0	0	0	1	11	66	n
e	11	0	0	0	0	0	0	0	0	0	0	0	1	12	e
s	12	0	0	0	0	0	0	0	0	0	0	0	0	1	s
	*	0	1	3	7	15	31	63	127	255	511	1023	2047	4095	
	+	1	2	4	8	16	32	64	128	256	512	1024	2048	4096	